

# 第九章

## 重积分

### 同步练习

## 9.2.一

1. 三次积分  $\int_0^1 dx \int_0^x dy \int_0^y dz =$  \_\_\_\_\_

$$\begin{aligned}\text{解} \quad \int_0^1 dx \int_0^x dy \int_0^y dz &= \int_0^1 dx \int_0^x [z]_0^y dy \\ &= \int_0^1 dx \int_0^x y dy \\ &= \int_0^1 \left[ \frac{1}{2} y^2 \right]_0^x dx \\ &= \frac{1}{2} \int_0^1 x^2 dx \\ &= \frac{1}{2} \left[ \frac{1}{3} x^3 \right]_0^1 \\ &= \frac{1}{6}\end{aligned}$$

## 9.2.一

2. 设 $\Omega$ 由 $x^2 + y^2 = 2z$ 及 $z = 2$ , 则 $\iiint_{\Omega} z dv = \underline{\hspace{2cm}}$

解  $\Omega$ 是 $Z$ -型区域 先二后一

$$0 \leq z \leq 2 \quad D_z = \{(x, y) \mid x^2 + y^2 \leq 2z\}$$

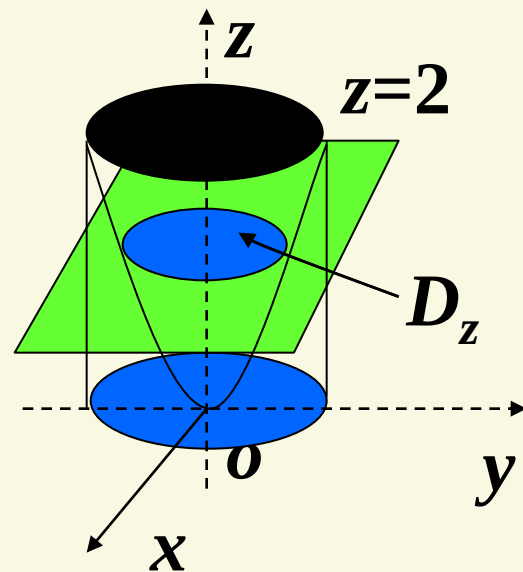
$$\iiint_{\Omega} z dv = \int_0^2 z dz \iint_{x^2+y^2 \leq 2z} dx dy$$

$$= \int_0^2 z \cdot S_D dz$$

$$= \int_0^2 z \pi \cdot 2z dz$$

$$= \pi \left[ \frac{2}{3} z^3 \right]_0^2$$

$$= \frac{16}{3} \pi$$



## 9.2.一

3. 设  $\Omega = \{(x, y, z) \mid x^2 + y^2 \leq 1, 0 \leq z \leq 2\}$ ,

则  $\iiint_{\Omega} \sqrt{x^2 + y^2} dv = \underline{\hspace{2cm}}$

解  $\Omega$  是  $Z$ -型区域 先二后一

$$\iiint_{\Omega} \sqrt{x^2 + y^2} dv = \int_0^2 dz \iint_{x^2 + y^2 \leq 1} \sqrt{x^2 + y^2} dx dy$$

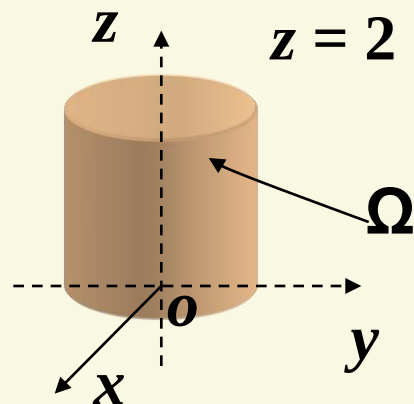
$$= \int_0^2 z dz \iint_{r \leq 1} r \cdot r dr d\theta$$

$$= \int_0^2 z dz \int_0^{2\pi} d\theta \int_0^1 r^2 dr$$

$$= \int_0^2 z dz \int_0^{2\pi} \left[ \frac{1}{3} r^3 \right]_0^1 d\theta$$

$$= \frac{1}{3} \int_0^2 z dz \int_0^{2\pi} d\theta = \frac{1}{3} \int_0^2 z [\theta]_0^{2\pi} dz$$

$$= \frac{2\pi}{3} \int_0^2 z dz = \frac{2\pi}{3} \left[ \frac{1}{2} z^2 \right]_0^2 = \frac{4\pi}{3}$$



## 9.2.一

4. 设  $\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$ ,

则  $\iiint_{\Omega} (x^2 + y^2 + z^2) dv = \underline{\hspace{2cm}}$

解  $\Omega$  是  $Z$ -型区域      球面坐标系

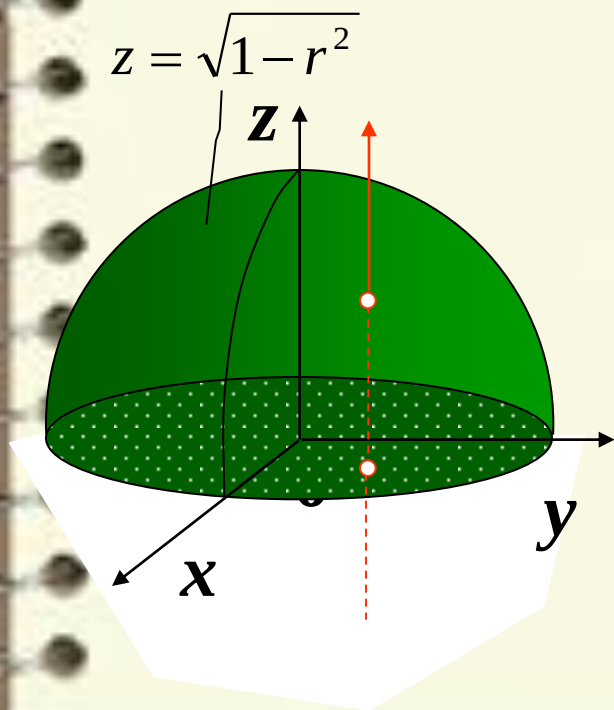
$$\iiint_{\Omega} (x^2 + y^2 + z^2) dv = \iiint_{\rho \leq 1} \rho^2 \cdot \rho^2 \sin \varphi d\rho d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 \rho^4 \sin \varphi d\rho$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin \varphi \left[ \frac{1}{5} \rho^5 \right]_0^1 d\varphi$$

$$= \frac{1}{5} \int_0^{2\pi} d\theta \int_0^{\pi} \sin \varphi d\varphi = \frac{1}{5} \int_0^{2\pi} [-\cos \varphi]_0^{\pi} d\theta$$

$$= \frac{2}{5} \int_0^{2\pi} d\theta = \frac{2}{5} [\theta]_0^{2\pi} = \frac{4\pi}{5}$$



## 9.2.二

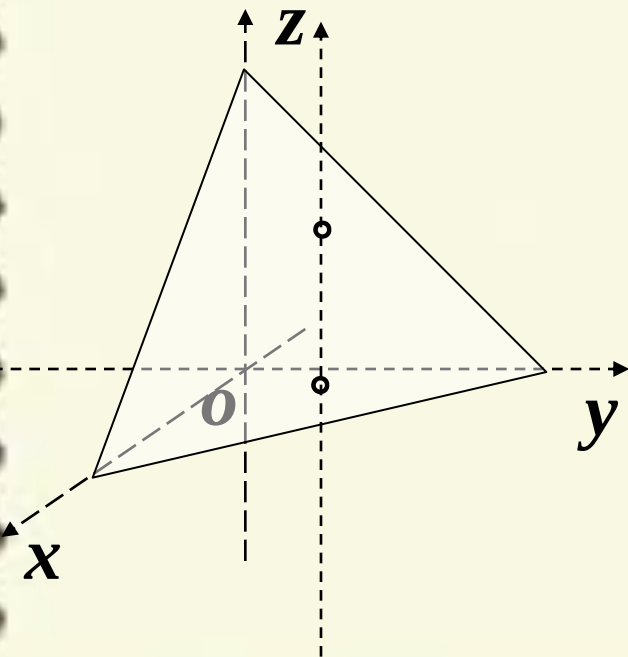
1. 求  $\iiint_{\Omega} \frac{1}{(1+x+y+z)^3} dv$ ,

式中  $\Omega$  由平面  $x+y+z=1$  与三个坐标平面围成的区域.

解:  $\Omega$  是 Z-型区域 先一后二

$$D_{xoy} = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1-x\}$$

$$\begin{aligned} \iiint_{\Omega} \frac{1}{(1+x+y+z)^3} dv &= \iint_D dx dy \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3} dz \\ &= \iint_D \left[ -\frac{1}{8} + \frac{1}{2(1+x+y)^2} \right] dx dy \\ &= \int_0^1 dx \int_0^{1-x} \left[ -\frac{1}{8} + \frac{1}{2(1+x+y)^2} \right] dy \\ &= -\frac{1}{8} \int_0^1 (1-x) dx - \frac{1}{2} \int_0^1 \left[ -\frac{1}{2} - \frac{1}{1+x} \right] dx \\ &= \frac{1}{2} \ln 2 - \frac{5}{16} \end{aligned}$$



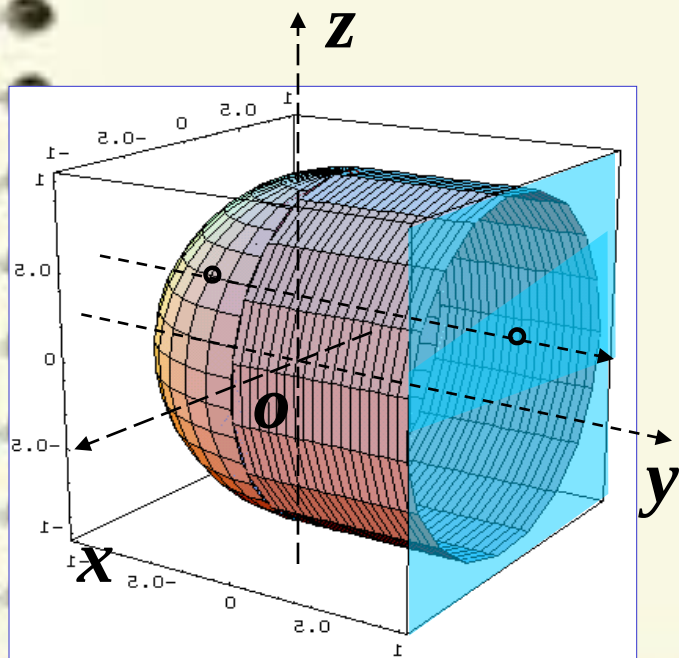
## 9.2.二

2. 求  $\iiint_{\Omega} y\sqrt{1-x^2} dv$ , 式中  $\Omega$  由曲面  $y = -\sqrt{1-x^2-z^2}$ ,  $x^2+z^2=1$  及平面  $y=1$  围成的区域

解  $\Omega$  是 Y-型区域 先一后二

$$D_{zox} = \{(x, z) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}\}$$

$$\begin{aligned} \iiint_{\Omega} y\sqrt{1-x^2} dv &= \iint_{D_{xz}} \sqrt{1-x^2} dx dz \int_{-\sqrt{1-x^2-z^2}}^1 y dy \\ &= \int_{-1}^1 \sqrt{1-x^2} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \int_{-\sqrt{1-x^2-z^2}}^1 y dy \\ &= \int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2} \frac{x^2+z^2}{2} dz \\ &= \frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} \cdot 2 \left( x^2 z + \frac{z^3}{3} \right) \Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx \\ &= \frac{1}{3} \int_{-1}^1 [3x^2(1-x^2) + (1-x^2)^2] dx \end{aligned}$$



## 9.2.二

2. 求  $\iiint_{\Omega} y\sqrt{1-x^2} dv$ , 式中  $\Omega$  由曲面  $y = -\sqrt{1-x^2-z^2}$ ,  $x^2+z^2=1$  及平面  $y=1$  围成的区域.

解  $\Omega$  是  $Y$ -型区域 先一后二

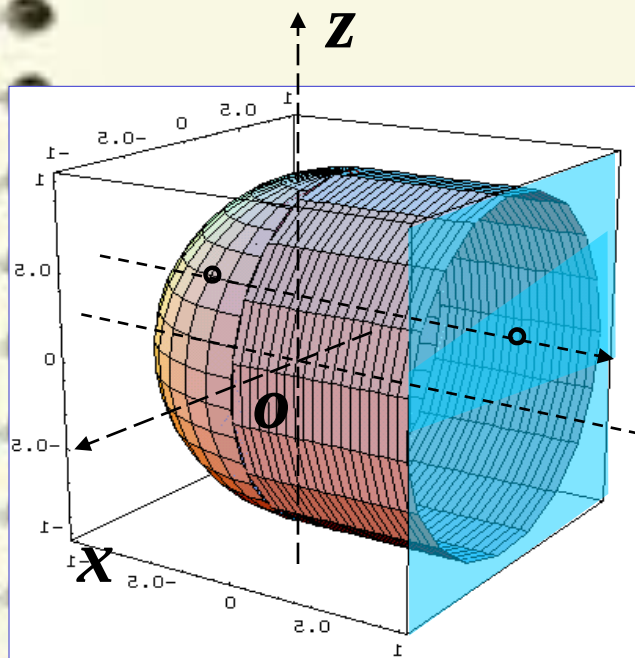
$$D_{zOx} = \{(x, z) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq z \leq \sqrt{1-x^2}\}$$

$$\iiint_{\Omega} y\sqrt{1-x^2} dv = \int_{-1}^1 \sqrt{1-x^2} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \int_{-\sqrt{1-x^2-z^2}}^1 y dy$$

$$= \frac{1}{3} \int_{-1}^1 [3x^2(1-x^2) + (1-x^2)^2] dx$$

$$= \frac{1}{3} \int_{-1}^1 (1+x^2-2x^4) dx$$

$$= \frac{2}{3} \left[ x + \frac{1}{3} x^3 - \frac{2}{5} x^5 \right]_0^1 = \frac{28}{45}.$$



## 9.2.二

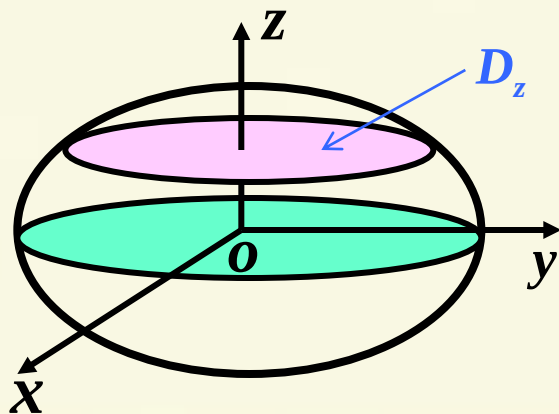
3. 求  $\iiint_{\Omega} z^2 dv$ , 式中  $\Omega$  由曲面  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$  围成的区域.

解  $\Omega$  是  $Z$ -型区域 先二后一

$$-c \leq z \leq c \quad D_z = \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 - \frac{z^2}{c^2}\}$$

$$\iint_{D_z} dx dy = \pi \sqrt{a^2(1 - \frac{z^2}{c^2})} \cdot \sqrt{b^2(1 - \frac{z^2}{c^2})} = \pi ab(1 - \frac{z^2}{c^2}),$$

$$\iiint_{\Omega} z^2 dv = \int_{-c}^c z^2 dz \iint_{D_z} dx dy = \int_{-c}^c \pi ab(1 - \frac{z^2}{c^2}) z^2 dz$$



$$= 2\pi ab \int_0^c (z^2 - \frac{z^4}{c^2}) dz$$

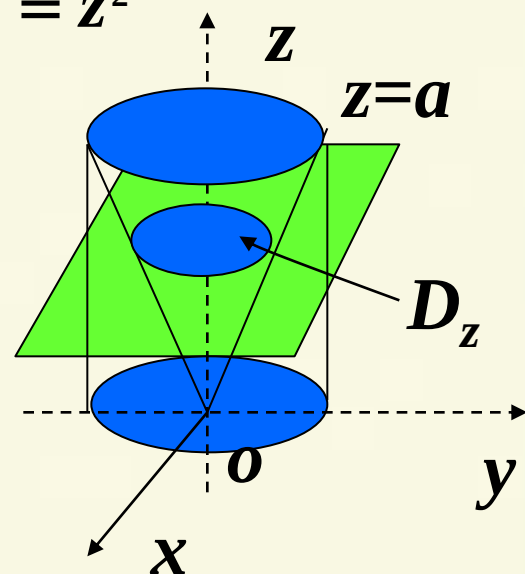
$$= 2\pi ab [\frac{1}{3} z^3 - \frac{1}{5} z^5]_0^c = \frac{4}{15} \pi abc^3$$

## 9.2.二

4. 求  $\iiint_{\Omega} (x^2 + y^2) dv$ , 式中  $\Omega$  由锥面  $x^2 + y^2 = z^2$  及平面  $z = a (a > 0)$  围成的区域.

解  $\Omega$  是  $Z$ -型区域 先二后一

$$0 \leq z \leq a \quad D_z = \{(x, y) \mid x^2 + y^2 \leq z^2\}$$



$$\iiint_{\Omega} (x^2 + y^2) dv = \int_0^a dz \iint_{D_z} (x^2 + y^2) dx dy$$

$$= \int_0^a dz \iint_{r \leq z} r^2 \cdot r dr d\theta$$

$$= \int_0^a dz \int_0^{2\pi} d\theta \int_0^z r^3 dr = \int_0^a dz \int_0^{2\pi} \left[ \frac{1}{4} r^4 \right]_0^z d\theta$$

$$= \frac{1}{4} \int_0^a z^4 dz \int_0^{2\pi} d\theta = \frac{\pi}{2} \int_0^a z^4 dz$$

$$= \frac{\pi}{10} [z^5]_0^a = \frac{\pi a^5}{10}$$

## 9.2.二

5. 求  $\iiint_{\Omega} r \cos \theta \, dr d\theta \, dz$ ,

式中  $\Omega = \{(\theta, r, z) \mid 0 \leq \theta \leq \frac{\pi}{2}, 1 \leq r \leq 2, 0 \leq z \leq 1\}$ .

解  $\iiint_{\Omega} r \cos \theta \, dr d\theta \, dz$

$$\begin{aligned} &= \int_0^{\frac{\pi}{2}} \cos \theta d\theta \int_1^2 r dr \int_0^1 dz \\ &= \int_0^{\frac{\pi}{2}} \cos \theta d\theta \int_1^2 r [z]_0^1 dr = \int_0^{\frac{\pi}{2}} \cos \theta d\theta \int_1^2 r dr \\ &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos \theta [r^2]_1^2 d\theta = \frac{3}{2} \int_0^{\frac{\pi}{2}} \cos \theta d\theta \\ &= \frac{3}{2} [\sin \theta]_0^{\frac{\pi}{2}} = \frac{3}{2} \end{aligned}$$

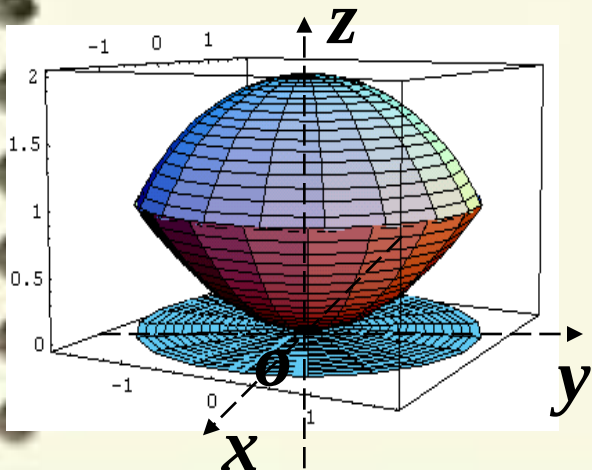
## 9.2.二

6. 求 $\iiint_{\Omega} z dv$ , 式中 $\Omega$ 由球面 $x^2 + y^2 + z^2 = 4$ 及旋转抛物面 $x^2 + y^2 = 3z$ 围成的区域.

解  $\Omega$ 是Z-型区域 先一后二

$$D_{xoy} = \{(x, z) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$$

$$\begin{aligned}\iiint_{\Omega} z dv &= \iint_{D_{xoy}} d\sigma \int_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} z dz = \iint_{D_{xoy}} \left[ \frac{1}{2} z^2 \right]_{\frac{x^2+y^2}{3}}^{\sqrt{4-x^2-y^2}} d\sigma \\ &= \frac{1}{2} \iint_{D_{xoy}} \left[ 4 - x^2 - y^2 - \frac{1}{9} (x^2 + y^2)^2 \right] d\sigma \\ &= \frac{1}{2} \iint_{r \leq 1} \left[ 4 - r^2 - \frac{1}{9} r^4 \right] r dr d\theta \\ &= \frac{1}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left( 4r - r^3 - \frac{r^5}{9} \right) dr \\ &= \frac{1}{2} [\theta]_0^{2\pi} \left[ r^2 - \frac{r^4}{4} - \frac{r^6}{54} \right]_0^{\sqrt{3}} = \frac{13}{4} \pi.\end{aligned}$$



## 9.2.二

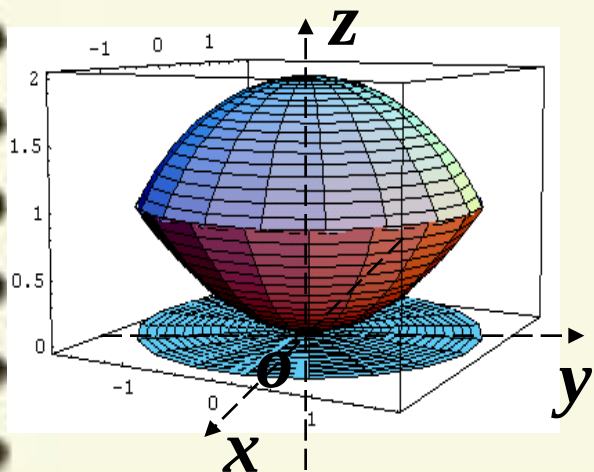
6. 求  $\iiint_{\Omega} z dv$ , 式中  $\Omega$  由球面  $x^2 + y^2 + z^2 = 4$  及旋转抛物面  $x^2 + y^2 = 3z$  围成的区域.

解  $\Omega$  是  $Z$ -型区域 先二后一

$$0 \leq z \leq 1 \quad D_{1z} = \{(x, y) \mid x^2 + y^2 \leq 3z\}$$

$$1 \leq z \leq 2 \quad D_{2z} = \{(x, y) \mid x^2 + y^2 \leq 4 - z^2\}$$

$$\iiint_{\Omega} z dv = \int_0^1 z dz \iint_{D_{1z}} dx dy + \int_1^2 z dz \iint_{D_{2z}} dx dy = \int_0^1 z S_{D_{1z}} dz + \int_1^2 z S_{D_{2z}} dz$$



$$= 3\pi \int_0^1 z^2 dz + \pi \int_1^2 z(4 - z^2) dz$$

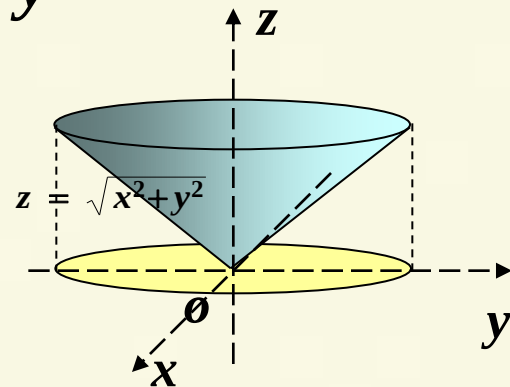
$$= \pi [z^3]_0^1 + \pi [2z^2 - \frac{1}{4} z^4]_1^2$$

$$= \pi + 6\pi - \frac{15}{4}\pi = \frac{13}{4}\pi.$$

## 9.2.二

7. 求  $\iiint_{\Omega} (y+z) dv$ , 式中  $\Omega$  由锥面  $z = \sqrt{x^2 + y^2}$  及平面  $z = 1$  围成的区域.

解  $\Omega$  是  $Z$ -型区域 先一后二



$$\iiint_{\Omega} y dv = \iint_{D_{zox}} dz dx \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} y dy = 0$$

$$\iiint_{\Omega} z dv = \int_0^1 z dz \iint_{D_z} dx dy \quad \text{先二后一}$$

$$= \int_0^1 z \pi z^2 dz = \pi \int_0^1 z^3 dz = \pi \left[ \frac{1}{4} z^4 \right]_0^1 = \frac{\pi}{4}$$

$$\iiint_{\Omega} (y+z) dv = 0 + \frac{\pi}{4} = \frac{\pi}{4}$$

## 9.2.二

8. 求  $\iiint_{\Omega} \sqrt{x^2 + y^2} dv$ ,

式中  $\Omega$  是由曲面  $x^2 + y^2 = 2z$  及平面  $z = 2$  围成的区域.

解  $\Omega$  是  $Z$ -型区域 先二后一

$$0 \leq z \leq 2 \quad D_z = \{(x, y) \mid x^2 + y^2 \leq 2z\}$$

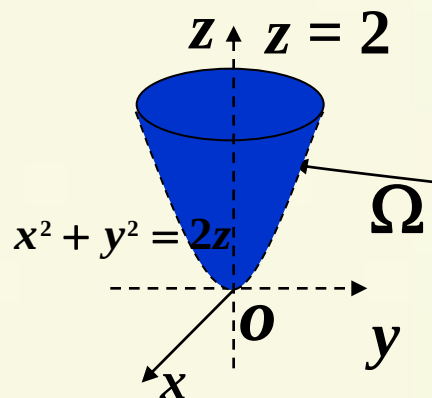
$$\iiint_{\Omega} \sqrt{x^2 + y^2} dv = \int_0^2 dz \iint_{D_z} \sqrt{x^2 + y^2} dx dy$$

$$= \int_0^2 dz \iint_{r^2 \leq 2z} r \cdot r dr d\theta$$

$$= \int_0^2 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{2z}} r^2 dr = \int_0^2 dz \int_0^{2\pi} \left[ \frac{1}{3} r^3 \right]_0^{\sqrt{2z}} d\theta$$

$$= \int_0^2 \frac{2\sqrt{2}}{3} z^{\frac{3}{2}} dz \int_0^{2\pi} d\theta = \frac{4\sqrt{2}}{3} \pi \int_0^2 z^{\frac{3}{2}} dz$$

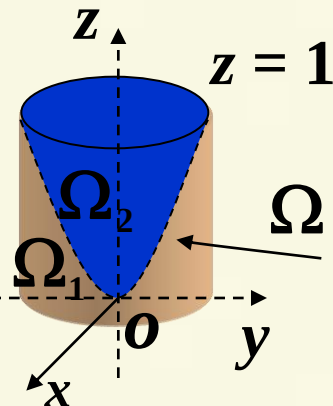
$$= \frac{4\sqrt{2}}{3} \pi \left[ \frac{2}{5} z^{\frac{5}{2}} \right]_0^2 = \frac{64}{15} \pi$$



## 9.2.二

9.求 $\iiint_{\Omega} |z - x^2 - y^2| dv$ , 式中 $\Omega = \{(x, y, z) \mid x^2 + y^2 \leq 1, 0 \leq z \leq 1\}$ .

解  $\Omega$ 是Z-型区域 先一后二



$$\iiint_{\Omega} |z - x^2 - y^2| dv = \iiint_{\Omega_1} (x^2 + y^2 - z) dv + \iiint_{\Omega_2} (z - x^2 - y^2) dv$$

$$= \iiint_{\substack{z \leq r^2 \\ r \leq 1}} (r^2 - z) r dr d\theta dz + \iiint_{\substack{r^2 \leq z \\ r \leq 1}} (z - r^2) r dr d\theta dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 r dr \int_0^{r^2} (r^2 - z) dz + \int_0^{2\pi} d\theta \int_0^1 r dr \int_{r^2}^1 (z - r^2) dz$$

$$= \int_0^{2\pi} d\theta \int_0^1 r \left[ r^2 z - \frac{1}{2} z^2 \right]_0^{r^2} dr + \int_0^{2\pi} d\theta \int_0^1 r \left[ \frac{1}{2} z^2 - r^2 z \right]_{r^2}^1 dr$$

$$= \int_0^{2\pi} d\theta \int_0^1 r \left( r^4 - \frac{1}{2} r^2 \right) dr + \int_0^{2\pi} d\theta \int_0^1 r \left( \frac{1}{2} - r^2 - \frac{1}{2} r^4 + r^4 \right) dr$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left( r^5 + \frac{1}{2} r - r^3 \right) dr = [\theta]_0^{2\pi} \left[ \frac{1}{6} r^6 + \frac{1}{4} r^2 - \frac{1}{4} r^4 \right]_0^1 = \frac{\pi}{3}$$

## 9.2.二

10. 求  $\iiint_{\Omega} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dv$ ,

式中  $\Omega$  由球面  $x^2 + y^2 + z^2 = 2z$  围成的区域.

解 球面坐标系

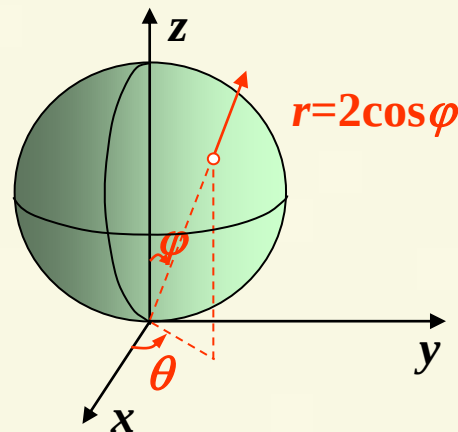
$$\iiint_{\Omega} \frac{1}{\sqrt{x^2 + y^2 + z^2}} dv = \iiint_{\Omega} \frac{1}{r} \cdot r^2 \sin \varphi dr d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r \sin \varphi dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \left[ \frac{1}{2} r^2 \right]_0^{2\cos\varphi} d\varphi$$

$$= 2 \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin \varphi \cos^2 \varphi d\varphi = 2 \int_0^{2\pi} \left[ -\frac{1}{3} \cos^3 \varphi \right]_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{2}{3} \int_0^{2\pi} d\theta = \frac{4}{3} \pi$$



## 9.2.二

11. 求  $\iiint_{\Omega} z \sqrt{x^2 + y^2 + z^2} dv$ ,

式中  $\Omega = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1, z \geq \sqrt{3(x^2 + y^2)}\}$ .

解 球面坐标系

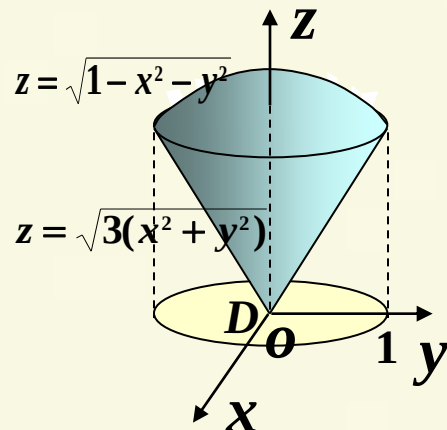
$$\iiint_{\Omega} z \sqrt{x^2 + y^2 + z^2} dv$$

$$= \iiint_{\Omega} r \cos \varphi \cdot r \cdot r^2 \sin \varphi dr d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} d\varphi \int_0^1 r^4 \sin \varphi \cos \varphi dr$$

$$= \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \sin \varphi \cos \varphi \left[ \frac{1}{5} r^5 \right]_0^1 d\varphi = \frac{1}{5} \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{6}} \sin \varphi \cos \varphi d\varphi$$

$$= \frac{1}{5} \int_0^{2\pi} \left[ \frac{1}{2} \sin^2 \varphi \right]_0^{\frac{\pi}{6}} d\theta = \frac{1}{40} \int_0^{2\pi} d\theta = \frac{\pi}{20}$$



## 9.2.二

12. 求由曲面 $z = x^2 + 2y^2$ 与 $z = 6 - 2x^2 - y^2$ 围成的立体体积

解  $\Omega$ 是Z-型区域 先一后二

$$\begin{cases} z = x^2 + 2y^2 \\ z = 6 - 2x^2 - y^2 \end{cases} \quad \text{得投影柱面: } x^2 + y^2 = 2 \quad D_{xoy}$$

$$V = \iiint_{\Omega} dv = \iint_{D_{xoy}} dx dy \int_{x^2+2y^2}^{6-2x^2-y^2} dz$$

$$= \iint_{D_{xoy}} (6 - 2x^2 - y^2 - x^2 - 2y^2) dx dy$$

$$= 3 \iint_{D_{xoy}} (2 - x^2 - y^2) dx dy \quad \text{极坐标系}$$

$$= 3 \iint_{r \leq \sqrt{2}} (2 - r^2) r dr d\theta = 3 \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (2r - r^3) dr$$

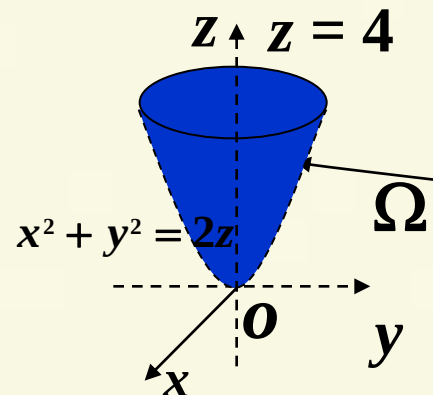
$$= 3 \int_0^{2\pi} \left[ r^2 - \frac{1}{4} r^4 \right]_0^{\sqrt{2}} d\theta = 3 \int_0^{2\pi} d\theta = 6\pi$$

## 9.2.二

13. 求  $\iiint_{\Omega} (x^2 + y^2 + z^2) dv$ , 式中  $\Omega$  由曲线  $\begin{cases} y^2 = 2z \\ x = 0 \end{cases}$  绕  $z$  轴  
旋转一周而成曲面与平面  $z = 4$  围成的区域

解  $\Omega$  是  $Z$ -型区域 先二后一

$$0 \leq z \leq 4 \quad D_z = \{(x, y) \mid x^2 + y^2 \leq 2z\}$$



$$\iiint_{\Omega} (x^2 + y^2 + z^2) dv$$

$$= \int_0^4 dz \iint_{D_z} (x^2 + y^2 + z^2) dx dy = \int_0^4 dz \iint_{r^2 \leq 2z} (r^2 + z^2) \cdot r dr d\theta$$

$$= \int_0^4 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{2z}} (r^3 + z^2 r) dr = \int_0^4 dz \int_0^{2\pi} \left[ \frac{1}{4} r^4 + \frac{1}{2} z^2 r^2 \right]_0^{\sqrt{2z}} d\theta$$

$$= \int_0^4 (z^2 + z^3) dz \int_0^{2\pi} d\theta = 2\pi \int_0^4 (z^2 + z^3) dz$$

$$= 2\pi \left[ \frac{1}{3} z^3 + \frac{1}{4} z^4 \right]_0^4 = \frac{512}{3} \pi$$

### 9.2.三

证明题 设 $f(x)$ 是连续函数且 $f(0)=0, f'(0)=2$ , 证明:

$$\lim_{t \rightarrow 0} \frac{1}{t^4} \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dv = 2\pi$$

$$\text{证} \quad \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dv = \iiint_{r \leq t} f(r) r^2 \sin \varphi dr d\varphi d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^\pi d\varphi \int_0^t f(r) r^2 \sin \varphi dr = \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^t f(r) r^2 dr$$

$$= \left[ \int_0^t f(r) r^2 dr \right] \int_0^{2\pi} [-\cos \varphi]_0^\pi d\theta = 4\pi \int_0^t f(r) r^2 dr$$

$$\lim_{t \rightarrow 0} \frac{1}{t^4} \iiint_{x^2+y^2+z^2 \leq t^2} f(\sqrt{x^2+y^2+z^2}) dv = \lim_{t \rightarrow 0} \frac{1}{t^4} 4\pi \int_0^t f(r) r^2 dr$$

$$= 4\pi \lim_{t \rightarrow 0} \frac{1}{4t^3} f(t) t^2 = \pi \lim_{t \rightarrow 0} \frac{f(t)}{t} = \pi \lim_{t \rightarrow 0} \frac{f(t) - f(0)}{t}$$

$$= \pi f'(0) = 2\pi$$